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Non-Gibbsian Limit for Large-Block Majority-Spin Transformations

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We generalize a result of Lebowitz and Maes, that projections of massless Gaussian measures onto Ising spin configurations are non-Gibbs measures. This result provides the first evidence for the existence of singularities in majority-spin transformations of critical models. Indeed, under the assumption of the folk theorem that an average-block-spin transformation applied to a critical Ising model in 5 or more dimensions converges to a Gaussian fixed point, we show that the limit of a sequence of majority-spin transformations with increasing block size applied to a critical Ising model is a measure that is not of Gibbsian type.

KEY WORDS: Non-Gibbs measure; real-space renormalization.

1. INTRODUCTION

Recently, Lebowitz and Maes,[1] in a study of entropic repulsion of a surface by a wall, noticed that the projection onto Ising configurations of a critical nearest-neighbor Gaussian measure is not a Gibbs measure for any reasonable potential. Here we extend this result to include self-similar Gaussian models.[2-5] These are fixed points of an average-block-spin transformation, and they are supposed to describe the fluctuations of block spins at the critical point of an Ising model in sufficiently high dimensions ($d \geqslant 5$).[6-9]

For Ising models, the average-block-spin transformation $\mathscr{A}_L$ followed by a projection onto Ising configurations is identical to the majority-rule transformation $\mathscr{R}_L$ introduced by Niemeijer and van Leeuwen.[10] Now

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consider applying the average-block-spin transformation, using larger and larger blocks ($L \to \infty$), to a critical Ising model in dimension $d \geqslant 5$. The resulting measures are believed to converge to a self-similar Gaussian random field. It follows, by a simple continuity argument, that the images of the critical Ising model under the majority rule transformation converge as $L \to \infty$ to the projection onto Ising configurations of the self-similar Gaussian random field. But this latter measure is not Gibbsian. Our generalization of the Lebowitz–Maes theorem thus provides the first example of a *critical* model for which a sequence of renormalization transformations has a limit in the space of measures on Ising configurations, but not in the space of Ising Hamiltonians.

Let us emphasize that this sequence of transformations is *not* the same as the repeated application of a majority transformation with fixed block size. This is a consequence of the fact that the majority transformation is nonlinear in the spin variables, so that $\mathscr{R}_L \mathscr{R}_{L'} \neq \mathscr{R}_{LL'}$. Rather, this sequence belongs to the class of "large-cell RG" transformations as described in refs. 11–14. But such "large-cell" sequences are clearly part of the "renormalization group enterprise," broadly conceived, even though they do not have a semigroup structure. Our example does *not* rule out the possibility that the majority-rule RG map is well defined in the neighborhood of the critical point, but is does show that this neighborhood must shrink as the block size is increased. This casts doubt on the feasibility of a mathematical justification of the majority-type renormalization group procedure.

Doubts about discrete-spin transformations were already expressed by Wilson.[15] (Note that the average-block-spin transformation has a continuous-spin fixed point which exhibits no peculiarity.) The possible occurrence of peculiarities in real-space renormalization transformations has been studied extensively by Griffiths and Pearce[16–19] (see also refs. 11 and 20–22). They found several examples where the transformation appears to be ill defined or singular as a mapping of a space of Ising Hamiltonians into itself. The precise nature of the peculiarities they found was left open. Israel[20] considered the decimation transformation and showed that, in the low-temperature region, the transformed measure is not Gibbs. Decimation, however, cannot have a (nontrivial) fixed-point measure, in contrast to majority-type transformations. We remark that a measure which is not Gibbs can still be an equilibrium state in the sense of the variational principle for some many-spin interaction, but, in contrast to the case of Gibbs states,[23] the choice of interaction in this case is highly nonunique.[24–26]

Burkhardt and van Leeuwen[27] pointed out that the actual singularities had all been found at some distance from the critical point and that "no compelling evidence for singularities in the critical region ... has yet been found" (ref. 27, p. 17). Our observation shows, however, that the region of peculiarities might extend to the critical point in the limit of larger and larger blocks. This supports a conjecture made by Israel.[20] Recently additional evidence for peculiarities in the whole coexistence region was found by Hasenfrantz *et al.*[28,29] Their results include the case of linear transformations, but are complementary to ours in the sense that they do not treat critical models.

2. RESULTS AND PROOFS

The model we consider is a Gaussian self-similar random field on $\mathbb{Z}^d$ which has a covariance given by[2–5]

$$\mu_C(S_x S_y) = C_{xy} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \hat{c}(p) e^{ip(x-y)}\, dp \tag{1}$$

with

$$\hat{c}(p) = \sum_{k \in \mathbb{Z}^d} |p + 2\pi k|^{-2+\eta} \prod_{i=1}^{d} \frac{4\sin^2(p_i/2)}{(p_i + 2\pi k_i)^2} \tag{2}$$

and $0 \leqslant \eta < 1$. The Gaussian measure μ_C with covariance C and mean zero is well defined as a Radon measure on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ (with the topology of pointwise convergence) if $d \geqslant 3$, and the following result is well known.[5]

Proposition 1. μ_C is a Gibbs measure for the Gaussian interaction $\mathscr{V} = \{\mathscr{V}_X \mid X \subset \mathbb{Z}^d \text{ finite}\}$ given by

$$\mathscr{V}_{\{x\}}(S_x) = \tfrac{1}{2} B_{xx} S_x^2$$
$$\mathscr{V}_{\{x,y\}}(S_x, S_y) = B_{xy} S_x S_y$$
$$\mathscr{V}_X = 0 \qquad \text{if} \quad |X| > 2$$

where

$$B_{xy} = \frac{1}{(2\pi)^d} \int dp\; \hat{c}(p)^{-1} e^{ip(x-y)}$$

is the inverse of the covariance matrix C, and $d \geqslant 3$.

Formally, we can write the Hamiltonian for the Gaussian model as

$$H = \tfrac{1}{2} \sum_{x,y \in \mathbb{Z}^d} B_{xy} S_x S_y \tag{3}$$

From Eq. (2) it follows that $\hat{c}(p)^{-1} \in \mathscr{L}^\infty([-\pi,\pi]^d)$; $\hat{c}(-p) = \hat{c}(p)$ and $\hat{c}(p) \geqslant 0$. Furthermore,

$$B_{xy} \sim |x-y|^{-d-2+\eta} \tag{4}$$

so that

$$\sum_{y} |B_{xy}| < \infty \tag{5}$$

For further details see refs. 2–4. Essential for the Lebowitz–Maes theorem is that H is a critical (massless) Hamiltonian, i.e.,

$$\sum_{y \in \mathbb{Z}^d} B_{xy} = 0 \tag{6}$$

Hence there is a noncompact symmetry in the model, namely shifting the spins by a unform constant:

Proposition 2. For $a \in \mathbb{R}$, let μ_C^a be the Gaussian measure with mean a and covariance C defined by (1), i.e., $\mu_C^a = \tau_a \mu_C$, where τ_a shifts all spins by the constant a. Formally, τ_a is defined by

$$\int F(\{S_x\})\,(\tau_a \mu)(dS) = \int F(\{S_x + a\})\,\mu(dS)$$

for any integrable function F of the spins $\{S_x\}$. Then, for each $a \in \mathbb{R}$, μ_C^a is a Gibbs measure for the Hamiltonian (3).

Proof. Let $\mu_C^a(dS_\Lambda \mid \tilde{S}_{\Lambda^c})$ denote the conditional distribution of the spins S_Λ is a finite subset $\Lambda \subset \mathbb{Z}^d$, given a configuration of spins $\tilde{S}_{\Lambda^c}$ on the complement. The conditional distributions of μ_C^a are related to those of μ_C by

$$\mu_C^a(dS_\Lambda \mid \tilde{S}_{\Lambda^c}) = \tau_a \mu_C(dS_\Lambda \mid \tilde{S}_{\Lambda^c} - a)$$

for all finite $\Lambda \subset \mathbb{Z}^d$. Consequently,

$$\int F(S_\Lambda)\,\mu_C^a(dS_\Lambda \mid \tilde{S}_{\Lambda^c}) = \frac{\int F(S_\Lambda + a) \exp[-H(S_\Lambda \mid \tilde{S}_{\Lambda^c} - a)]\, dS_\Lambda}{\int \exp[-H(S_\Lambda \mid \tilde{S}_{\Lambda^c} - a)]\, dS_\Lambda}$$

where

$$H(S_\Lambda \mid S_{\Lambda^c}) = \sum_{X \cap \Lambda \neq \varnothing} \mathscr{V}_X(S_X)$$

If follows that

$$\begin{aligned} H(S_\Lambda \mid \tilde{S}_{\Lambda^c} - a) &= H(S_\Lambda \mid \tilde{S}_{\Lambda^c}) - a \sum_{x \in \Lambda} \sum_{y \in \Lambda^c} B_{xy} S_x \\ &= H(S_\Lambda \mid \tilde{S}_{\Lambda^c}) + a \sum_{x, y \in \Lambda} B_{xy} S_x \\ &= H(S_\Lambda + a \mid \tilde{S}_{\Lambda^c}) - a \sum_{x \in \Lambda} \sum_{y \in \Lambda^c} B_{xy} \tilde{S}_y - \tfrac{1}{2} a^2 \sum_{x, y \in \Lambda} B_{xy} \end{aligned} \tag{7}$$

so that

$$\int F(S_\Lambda)\,\mu_C^a(dS_\Lambda \mid \tilde{S}_{\Lambda^c}) = \frac{\int F(S_\Lambda)\exp[-H(S_\Lambda \mid \tilde{S}_{\Lambda^c})]\,dS_\Lambda}{\int \exp[-H(S_\Lambda \mid \tilde{S}_{\Lambda^c})]\,dS_\Lambda}$$

i.e., μ_C^a is also a Gibbs state. ▮

For each integer $L \geqslant 2$, consider the block-spin transformation

$$S'_x = L^{-(d+2-\eta)/2} \sum_{y \in B_L(x)} S_y \tag{8}$$

where the blocks $B_L(x)$ are hypercubes about Lx with sides of length L. This transformation on $\mathbb{R}^{\mathbb{Z}^d}$ induces a transformation $\mathscr{A}_L$ on the space $\mathscr{M}_I(\mathbb{R}^{\mathbb{Z}^d})$ of translation-invariant probability measures on $\mathbb{R}^{\mathbb{Z}^d}$. In particular, the measure μ_C is invariant under $\mathscr{A}_L$ for all L (that is why it is called "self-similar"). Moreover, in the case $\eta = 0$, μ_C is believed to be an attractive fixed point for the critical Ising model in dimension $d \geqslant 5$ ("non-central limit theorem"),[6-9] in the sense that

$$\lim_{L \to \infty} \mathscr{A}_L \mu_{\text{critical Ising}} = \mu_C \tag{9}$$

in the topology of weak convergence of measures. In other words, the distribution of properly rescaled block spins should approach the Gaussian fixed point μ_C as the block size L tends to infinity.

Now consider the projection map $\mathscr{P}\colon \mathscr{M}_I(\mathbb{R}^{\mathbb{Z}^d}) \to \mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d})$ induced by the transformation

$$\sigma_x = \operatorname{sgn} S_x \tag{10}$$

from continuous spins to discrete Ising spins. [The ambiguity of (10) at $S_x = 0$ is resolved by an arbitrary prescription; the simplest is to define σ_x to be $+1$ with probability 1/2 and to be -1 with probability 1/2.] Finally, let $\mathscr{I}\colon \mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d}) \to \mathscr{M}_I(\mathbb{R}^{\mathbb{Z}^d})$ be the trivial injection (every probability measure on $\{-1,1\}^{\mathbb{Z}^d}$ is also a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$). Then the Niemeijer–van Leeuwen[10] majority-rule transformation $\mathscr{R}_L$: $\mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d}) \to \mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d})$ is precisely

$$\mathscr{R}_L = \mathscr{P} \circ \mathscr{A}_L \circ \mathscr{I} \tag{11}$$

In other words, we have the following diagram:

$$\begin{array}{ccc}
\mathscr{M}_I(\mathbb{R}^{\mathbb{Z}^d}) & \xrightarrow{\ \mathscr{A}_L\ } & \mathscr{M}_I(\mathbb{R}^{\mathbb{Z}^d}) \\
\Big\uparrow{\scriptstyle \mathscr{I}} & & \Big\downarrow{\scriptstyle \mathscr{P}} \\
\mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d}) & \xrightarrow{\ \mathscr{R}_L\ } & \mathscr{M}_I(\{-1,1\}^{\mathbb{Z}^d})
\end{array}$$

Now consider applying the Niemeijer–van Leeuwen majority-rule transformation $\mathscr{R}_L$, on larger and larger blocks, to a critical Ising measure. We get

$$\begin{aligned}\lim_{L\to\infty} \mathscr{R}_L\, \mu_{\text{critical Ising}} &= \lim_{L\to\infty} \mathscr{P}\circ\mathscr{A}_L \mu_{\text{critical Ising}}\\ &= \mathscr{P}\big(\lim_{L\to\infty} \mathscr{A}_L \mu_{\text{critical Ising}}\big)\\ &= \mathscr{P}\mu_C\end{aligned}$$

where we have used (9) together with the continuity of $\mathscr{P}$ at μ_C. [Note that $\mathscr{P}$ is *not* a continuous map from $\mathscr{M}_t(\mathbb{R}^{\mathbb{Z}^d})$ to $\mathscr{M}(\{-1, 1\}^{\mathbb{Z}^d})$, because of the discontinuity of the sgn function at 0. But $\mathscr{P}$ is continuous at all measures $\mu\in\mathscr{M}_t(\mathbb{R}^{\mathbb{Z}^d})$ that give zero probability to the set $\{S\,|\,S_x=0 \text{ for some } x\}$. In particular, $\mathscr{P}$ is continuous at μ_C.] Proposition 4 below then implies that the limiting measure $\mathscr{P}\mu_C$ cannot be a Gibbs measure. This result follows from a generalization of the Lebowitz–Maes theorem:

Proposition 3. Let $\langle\cdot\rangle_a$ denote the expectation with respect to the measure μ_C^a and put $\sigma_x=\operatorname{sgn} S_x$. Then

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|}\ln\left\langle \exp\left(h\sum_{x\in\Lambda}\sigma_x\right)\right\rangle = |h|$$

Proof. We proceed as in ref. 1, Lemma 1. Thus we have, for $h>0$,

$$\begin{aligned}\exp(h|\Lambda|) &\geqslant \left\langle \exp h\sum_{x\in\Lambda}\sigma_x\right\rangle_0\\ &\geqslant \exp(h|\Lambda|)\left\langle \prod_{x\in\Lambda}\chi(S_x\geqslant 0)\right\rangle_0 \qquad (12)\end{aligned}$$

and

$$\left\langle \prod_{x\in\Lambda}\chi(S_x\geqslant 0)\right\rangle_0 \geqslant \left\langle \prod_{x\in\Lambda}\chi(|S_x|\leqslant k)\right\rangle_{-k} \qquad (13)$$

In the following we write $\chi_k(S_\Lambda)$ for $\prod_{x\in\Lambda}\chi(|S_x|\leqslant k)$. We estimate $\ln[\langle\chi_k(S_\Lambda)\rangle_{-k}/\langle\chi_k(S_\Lambda)\rangle_0]$ as follows: Writing $\mu_C^a(dS_{\Lambda^c})$ for the marginal distribution of the spins S_{Λ^c}, we have

$$\begin{aligned}\int F(S_\Lambda)\,\mu_C^a(dS) &= \int \mu_C^0(dS_{\Lambda^c})\,\frac{\int F(S_\Lambda)\exp[-H(S_\Lambda-a\,|\,S_{\Lambda^c})]\,dS_\Lambda}{\int\exp[-H(S_\Lambda\,|\,S_{\Lambda^c})]\,dS_\Lambda}\\ &= \langle F(S_\Lambda)\exp[-\tilde{H}_a(S)]\rangle_0 \qquad (14)\end{aligned}$$

with

$$\tilde{H}_a(S) = -a \sum_{x,\, y \in \Lambda} B_{xy} S_x + \tfrac{1}{2} a^2 \sum_{x,\, y \in \Lambda} B_{xy} - a \sum_{x \in \Lambda} \sum_{y \in \Lambda^c} B_{xy} S_y \tag{15}$$

In particular, $\langle \chi_k(S_\Lambda) \rangle_{-k} = \langle \chi_k(S_\Lambda) \exp[-\tilde{H}_{-k}(S)] \rangle_0$.

Using the Cauchy–Schwarz inequality, we find

$$\langle \chi_k(S_\Lambda) \rangle_0 \leqslant \langle \chi_k(S_\Lambda) \exp[-\tilde{H}_{-k}(S)] \rangle_0^{1/2} \langle \exp[\tilde{H} - k(S)] \rangle_0^{1/2} \tag{16}$$

and it follows that

$$\ln \langle \chi_k(S_\Lambda) \rangle_{-k} \geqslant 2 \ln \langle \chi_k(S_\Lambda) \rangle_0 - \ln \langle \exp[\tilde{H}_{-k}(S)] \rangle_0 \tag{17}$$

But

$$\ln \langle \exp[\tilde{H}_{-k}(S)] \rangle_0 = -k^2 \sum_{x \in \Lambda} \sum_{y \in \Lambda^c} B_{xy} \tag{18}$$

is of order $o(|\Lambda|)$ by (4). To bound the first term we use the Brascamp–Lieb inequalities as in ref. 1, Eq. (6). For the convenience of the reader we repeat this argument here. Define the expectation $\langle \cdot \rangle(k_\Lambda)$ by

$$\langle F(S) \rangle(k_\Lambda) = \frac{\langle F(S)\, \chi_k(S_\Lambda) \rangle_0}{\langle \chi_k(S_\Lambda) \rangle_0}$$

Then, if $y \in \Lambda$ and $\Lambda' = \Lambda \backslash \{y\}$,

$$\begin{aligned} \langle \chi_k(S_\Lambda) \rangle_0 &= [1 - \langle \chi(|S_y| > k) \rangle(k_{\Lambda'})] \langle \chi_k(S_{\Lambda'}) \rangle_0 \\ &\geqslant [1 - k^{-2} \langle S_y^2 \rangle(k_{\Lambda'})] \langle \chi_k(S_{\Lambda'}) \rangle_0 \end{aligned}$$

by Chebyshev's inequality. Now, by the Brascamp–Lieb inequalities,[30]

$$\langle S_y^2 \rangle \langle k_{\Lambda'}) \leqslant \langle S_y^2 \rangle_0 = C_{yy} \equiv c$$

Thus we find that

$$\langle \chi_k(S_\Lambda) \rangle_0 \geqslant \left(1 - \frac{c}{k^2}\right) \langle \chi_k(S_{\Lambda'}) \rangle_0$$

Iterating this inequality, we obtain

$$\ln \langle \chi_k(S_\Lambda) \rangle_0 \geqslant |\Lambda| \ln \left(1 - \frac{c}{k^2}\right) \tag{19}$$

(Note that we apply the BL inequalities to the marginal distribution of S_Λ, which is strictly positive.) We conclude that

$$\sup_k \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \left\langle \prod_{x \in \Lambda} \chi(|S_x| \leqslant k) \right\rangle_{-k} = 0 \tag{20}$$

This proves the proposition. ∎

Remark. In fact our proof only used that (a) there is a broken, noncompact symmetry for the Gibbs state, (b) the Brascamp–Lieb inequalities hold, and (c) the decay of the interaction satisfies condition (5).

Hence, our results extend to all models satisfying these conditions. In particular, they extend to all Gauss measures satisfying (5) and (6), e.g., with higher dimensional spins. It also holds in case the measure is not translationally invariant if we replace condition (5) by $\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} |B_{xy}| < \infty$.

From Proposition 3 and a result of Griffiths and Ruelle's[23] we may conclude that the induced measure $\tilde{\mu}$ on $\{-1, 1\}^{\mathbb{Z}^d}$ cannot be a Gibbs measure:

Proposition 4. Define $\int F(\sigma)\, \tilde{\mu}(d\sigma) = \int F(\operatorname{sgn} S)\, \mu_C(dS)$. Then there exists no Ising potential $\mathscr{V}$ such that $\sum_{X \ni 0} |\mathscr{V}(X)| < \infty$ for which $\tilde{\mu}$ is a Gibbs measure.

Proof. We shall prove that, if $\tilde{\mu}$ were a Gibbs measure with respect to $\mathscr{V}$, then the pressure $P(\mathscr{V} + h)$ would be piecewise linear in the external field h. This contradicts the result of ref. 23 that P is strictly convex. We have by definition

$$P(\mathscr{V}) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \sum_{\sigma_\Lambda} \exp\left[-\sum_{X \subset \Lambda} \mathscr{V}(X)\sigma_X \right] \tag{21}$$

and

$$\begin{aligned} P(\mathscr{V} + h) &= \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \sum_{\sigma_\Lambda} \exp\left[-\sum_{X \subset \Lambda} \mathscr{V}(X)\sigma_X - h \sum_{x \in \Lambda} \sigma_x \right] \\ &= P(\mathscr{V}) + \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \left\langle \exp\left(-h \sum_{x \in \Lambda} \sigma_x \right) \right\rangle_{\Lambda, \mathscr{V}} \end{aligned} \tag{22}$$

We now use the following result.

Lemma. The following relation holds:

$$\lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \left\langle \exp\left(-h \sum_{x \in \Lambda} \sigma_x \right) \right\rangle_{\Lambda, \mathscr{V}} = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln \left\langle \exp\left(-h \sum_{x \in \Lambda} \sigma_x \right) \right\rangle_0$$

It then follows from Proposition 3 that $P(\mathscr{V}+h)=P(\mathscr{V})-|h|$ (which is piecewise linear in h).

Proof of the Lemma. We have

$$\begin{aligned}
&\frac{1}{|\Lambda|}\ln\frac{\langle\exp(h\sum_{x\in\Lambda}\sigma_x)\rangle_0}{\langle\exp(h\sum_{x\in\Lambda}\sigma_x)\rangle_{\Lambda,\mathscr{V}}}\\
&\quad=\frac{1}{|\Lambda|}\ln\int\tilde{\mu}(d\sigma_{\Lambda^c})\\
&\qquad\times\frac{\sum_{\sigma_\Lambda}\exp[-H_{\mathscr{V}}(\sigma_\Lambda)+h\sum_{x\in\Lambda}\sigma_x+W_{\mathscr{V}}(\sigma_\Lambda\,|\,\sigma_{\Lambda^c})]}{\sum_{\sigma_\Lambda}\exp[-H_{\mathscr{V}}(\sigma_\Lambda)+W_{\mathscr{V}}(\sigma_\Lambda\,|\,\sigma_{\Lambda^c})]}\\
&\qquad\times\frac{\sum_{\sigma_\Lambda}\exp[-H_{\mathscr{V}}(\sigma_\Lambda)]}{\sum_{\sigma_\Lambda}\exp[-H_{\mathscr{V}}(\sigma_\Lambda)+h\sum_{x\in\Lambda}\sigma_x]}
\end{aligned}$$

As

$$|W_{\mathscr{V}}(\sigma_\Lambda\,|\,\sigma_{\Lambda^c})|\leqslant\sum_{\substack{X\cap\Lambda\neq\varnothing\\ X\cap\Lambda^c\neq\varnothing}}|\mathscr{V}(X)|$$

a standard argument as in ref. 24 or ref. 25 proves the lemma.[4]

3. CONCLUSIONS

The fact that Ising-to-Ising transformations, being nonlinear in the spin variables, may have problems of their own was already mentioned by Wilson.[(15)] The work of Griffiths and Pearce[(16–19)] has shown that these problems might in fact be serious because these renormalization transformations need not exist as maps of a reasonable space of Hamiltonians into itself. However, the singularities observed by these authors and others[(11,20–22,28,29)] were all situated in a region bounded away from the critical point. In fact, it was conjectured by Israel[(20)] that this singular region should close in on the critical point as the transformation is repeated. The example we have presented here supports this conjecture. We must stress that the majority transformation applied directly to a large block is not the same as a repeated transformation with a fixed block size. However, the former has also been used for investigating critical behavior. In view of the results of refs. 6–9 the block spin should have a Gaussian critical behavior in $d\geqslant 5$ dimensions. It is the projection of Gaussian models on Ising spins which causes the problems. In three dimensions the

[4] A different proof of Proposition 4 was kindly explained to us by Dr. C. Maes.

Gaussian self-similar process is not stable, and we do not know the stable fixed point for the transformation (8a). However, the Lebowitz–Maes theorem also applies to certain massless ϕ^4-models.[1] We therefore expect the same problems to arise for lower dimensional models.

It has been shown that real-space renormalization is mathematically well defined at very high temperatures or low densities.[11,16–21,31–33] Although our example does not rule out the possibility that it is also well defined in the neighborhood of the critical point, it does show that this neighborhood must shrink as the block size is increased. This casts doubt on the feasibility of a mathematical justification of the majority-type renormalization group procedure.

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